Bound states in a box

Sebastian König

in collaboration with D. Lee, H.-W. Hammer; S. Bour, U.-G. Meißner

Dr. Klaus Erkelenz Preis – Kolloquium

HISKP, Universität Bonn

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Past and present future

The Bonn potential – K. Erkelenz *et al.*

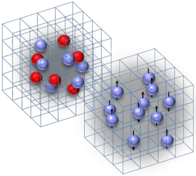
- based on field theoretical approach
- designed to be used in nuclear structure calculations

Chiral effective interactions

Modern structure calculations

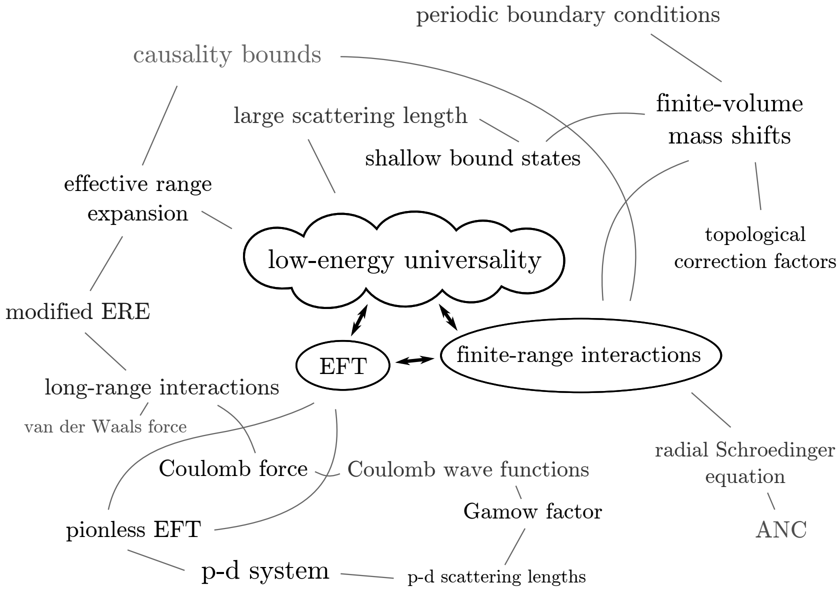
Nuclear Lattice

- No-core shell model
- Coupled cluster etc.
- SRG-evolved interactions
- ...



Epelbaum, Krebs, Lähde, Lee, Meißner

Outline


periodic boundary conditions
causality bounds
large scattering length
finite-volume mass shifts
shallow bound states
effective range expansion
topological correction factors
low-energy universality
modified ERE
EFT
finite-range interactions
long-range interactions
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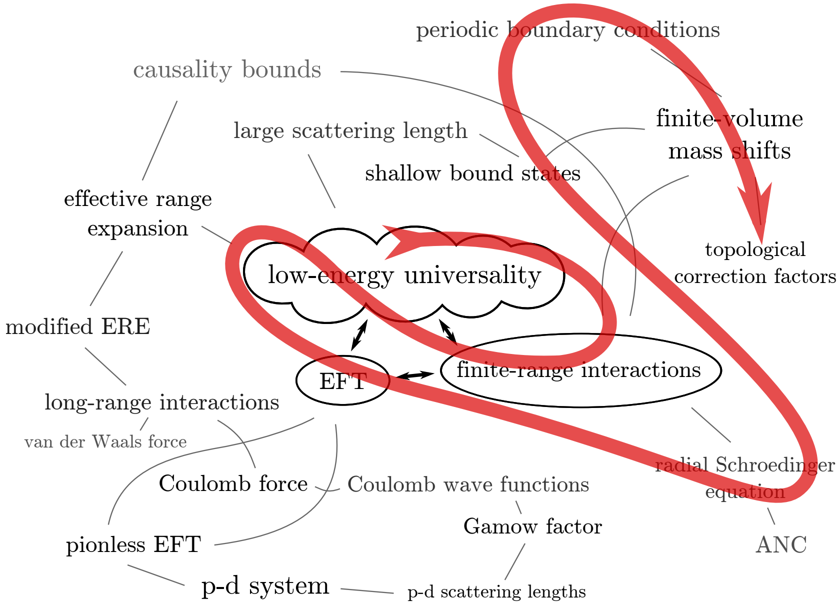

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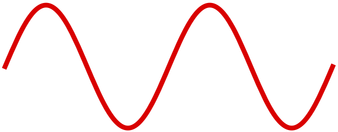
Prelude

Low-energy universality
and finite-range interactions

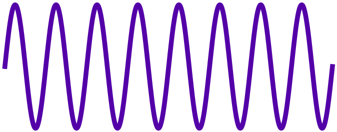
Low-energy universality

low energy $\rightarrow$ large wavelength



$\hookrightarrow$ **low resolution**

high energy $\rightarrow$ short wavelength



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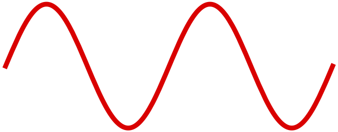

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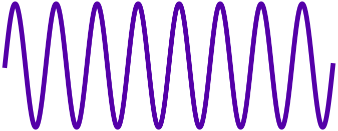

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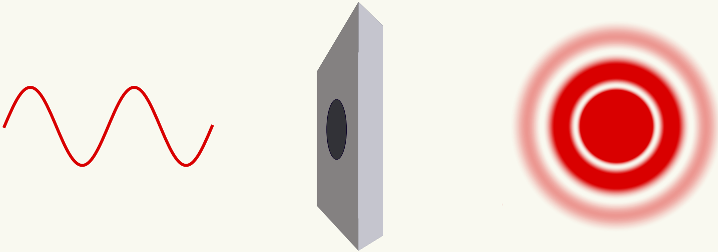
Resolution and diffraction



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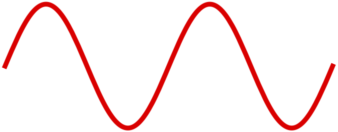

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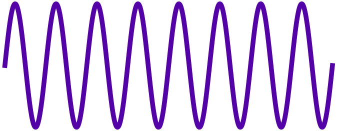

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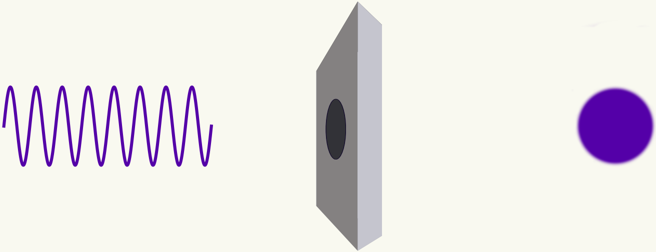

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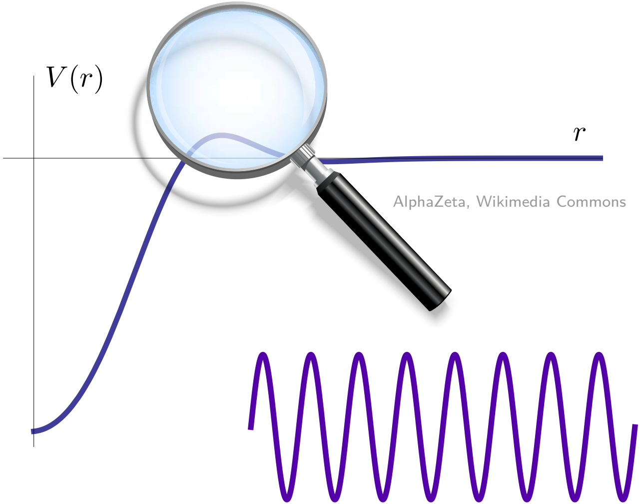

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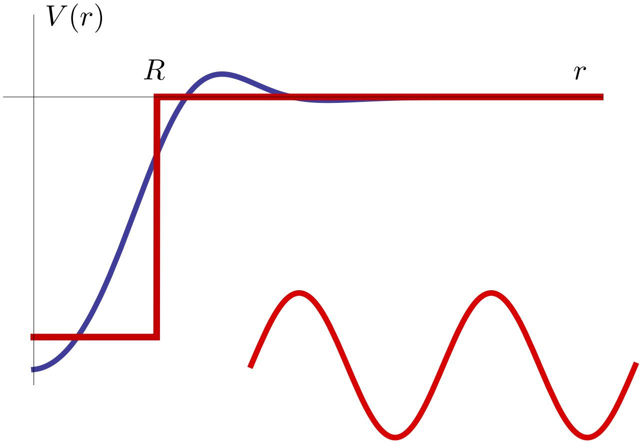
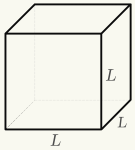


Bound states in a box

The box

- periodic finite volume
- cube of size L^3



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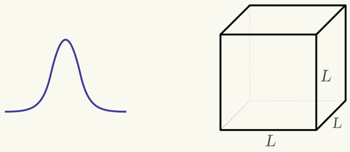
- 2-body bound states
- wavefunction ψ



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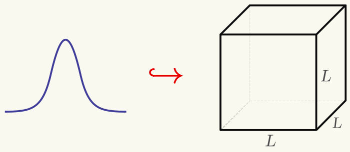

The finite volume changes the properties of the system!

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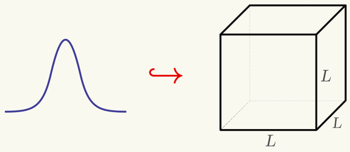


The finite volume changes the properties of the system!

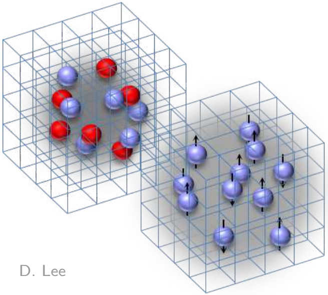
Important for numerical calculations $\rightarrow$ **Lattice**

Lattice calculations

Solve a physical theory by putting it on a spacetime-lattice!

Lattice QCD

- QCD observables from first principles
- quarks and gluons as degrees of freedom

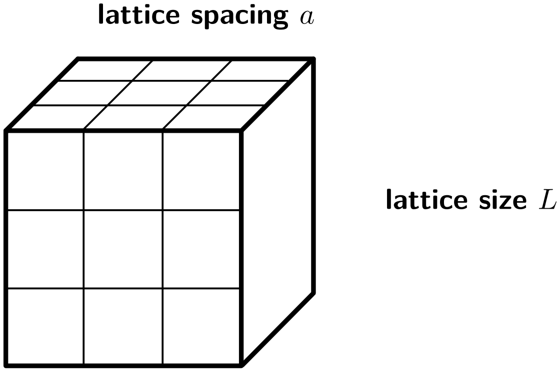


D. Lee

Nuclear Lattice Calculations

- nuclei from first principles
- nucleons and pions as d.o.f.
- based on chiral effective theory

Lattice artifacts



- $a \to 0$: continuum limit
- $L \to \infty$: infinite-volume limit

Lüscher's famous formula

Lüscher's idea

Use the volume dependence as a tool!

$$p \cot \delta_0(p) = \frac{1}{\pi L} S(\eta) \quad , \quad \eta = \left(\frac{Lp}{2\pi} \right)^2$$

$$p = p(E(L))$$

- measure energy levels in finite volume
- extract physical scattering phase shift

Outline

- **Overview**
- **Part I –**
  Mass shift of bound states with angular momentum
  arXiv:1103.4468, 1109.4577
- **Part II –**
  Topological factors in scattering systems
  arXiv:1107.1272
- **Summary**

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Part I

Mass shift of bound states with angular momentum

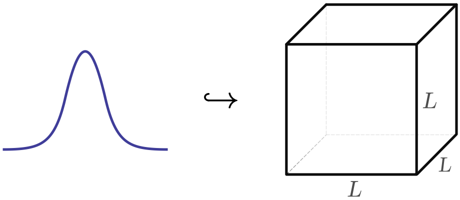
- **Lüscher's result for S-waves**
- **Bound states in a finite volume**
- **General result for arbitrary partial waves**
- **Sign of the mass shift**
- **Numerical tests**

Starting point

S-wave bound state

Lüscher (1986)

$$\Delta m_B = -24\pi |A|^2 \frac{\mathrm{e}^{-\kappa L}}{mL} + \mathcal{O}(\mathrm{e}^{-\sqrt{2}\kappa L})$$



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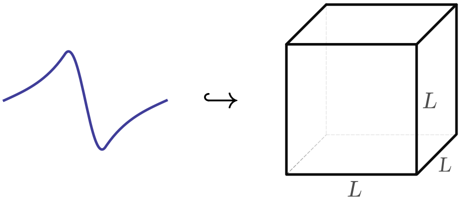
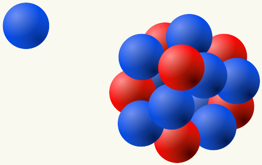


What's the the result for states with angular momentum?

Why care about higher partial waves?

Halo nuclei

- single nucleon weakly bound to a tight core



nucleus by Cam-Ann, Wikimedia Commons

Halo nuclei

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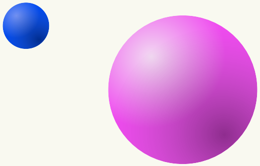
- can be described as an effective 2-body state



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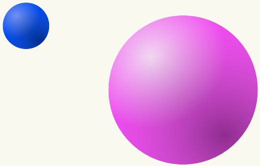

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Halo-EFT

expansion in $R_{\mathrm{core}}/R_{\mathrm{halo}} \to$ effective field theory

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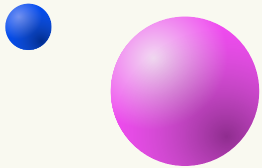

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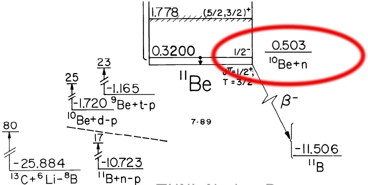


TUNL Nuclear Data

Example

P-wave state just below $^{10}\text{Be}+\text{n}$ threshold in ^{11}Be

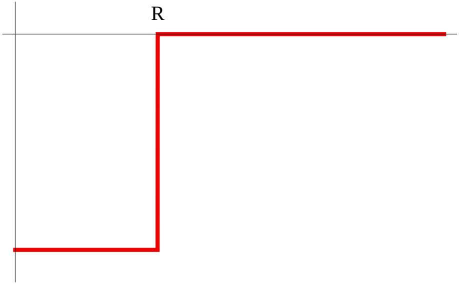
Schrödinger equation

$$\hat{H} = -\frac{1}{2\mu}\Delta_r + V(r)$$

$$\hat{H}\,|\psi_B\rangle = -\frac{\kappa^2}{2\mu}\,|\psi_B\rangle$$

finite-range interaction:

$V(r) = 0$ for $r > R$



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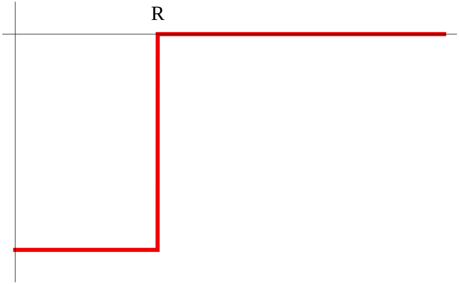

Radial Schrödinger equation

$$\psi_B(\mathbf{r}) = \frac{u_\ell(r)}{r} Y_\ell^m(\theta,\phi) \rightsquigarrow \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\ell(\ell+1)}{r^2} - 2\mu V(r) - \kappa^2\right) u_\ell(r) = 0$$

Asymptotic wavefunction

Radial Schrödinger equation

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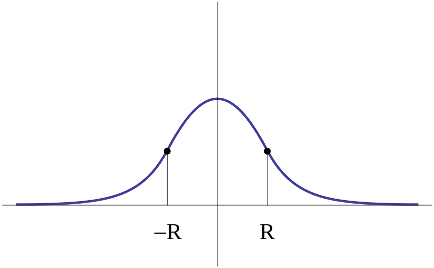

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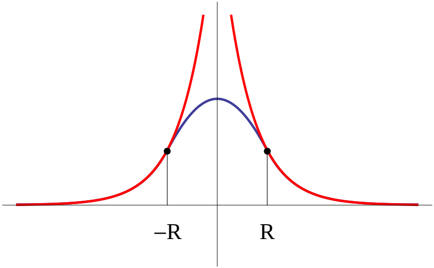
$$\rightsquigarrow u_\ell(r) = \mathrm{i}^\ell \gamma \, \hat{h}_\ell^+(i\kappa r)$$

Riccati–Hankel functions

$$\hat{h}_0^+(z) = \mathrm{e}^{\mathrm{i}z}$$

$$\hat{h}_1^+(z) = \left(1 + \frac{\mathrm{i}}{z}\right) \mathrm{e}^{\mathrm{i}(z - \pi/2)}$$

$$\hat{h}_2^+(z) = \cdots$$



Finite volume

Periodic boundary conditions

$\rightarrow$ infinitely many copies of the potential

$$V_L(\mathbf{r}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} V(\mathbf{r}+\mathbf{n}L) \quad , \quad L \gg R$$

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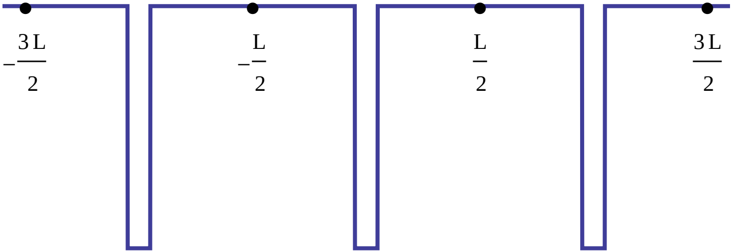

$$V(r) = V_0\,\theta(R-r)$$

Finite volume

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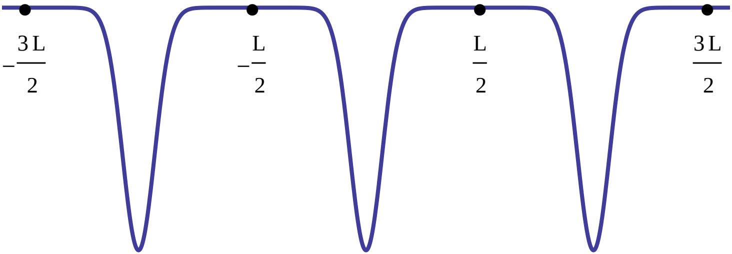

$$V(r) = V_0 \exp(-r^2/R^2)$$

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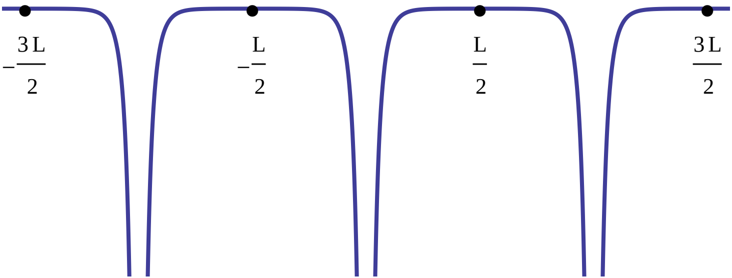

$$V(r) = V_0 \frac{\exp(-r/R)}{r}$$

Finite volume

$$\hat{H}_L \left|\psi\right\rangle = -E_B(L) \left|\psi\right\rangle$$
$$\hat{H} \left|\psi_B\right\rangle = -E_B(\infty) \left|\psi_B\right\rangle$$

Mass shift

$$\Delta m_B \equiv E_B(\infty) - E_B(L)$$
$$m_B = M - E_B$$

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The wavefunction $\psi(\mathbf{r})$ has to be periodic, too! $\boxed{\psi(\mathbf{r} + \mathbf{n}L) = \psi(\mathbf{r})}$

Ansatz: $\psi_0(\mathbf{r}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} \psi_B(\mathbf{r}+\mathbf{n}L)$

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$$\eta(\mathbf{r}) = \sum_{\mathbf{n}\neq\mathbf{n}'} V(\mathbf{r}+\mathbf{n}L)\,\psi_B(\mathbf{r}+\mathbf{n}'L)$$

$$\langle\psi|\,\hat{H}_L\,|\psi_0\rangle = -E_B(L)\,\langle\psi|\psi_0\rangle = -E_B(\infty)\,\langle\psi|\psi_0\rangle + \langle\psi|\eta\rangle$$

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Result

$$\Delta m_B = \frac{\langle\psi|\eta\rangle}{\langle\psi_0|\psi_0\rangle} = \sum_{|\mathbf{n}|=1} \int \mathrm{d}^3r\,\psi_B^*(\mathbf{r})\,V(\mathbf{r})\,\psi_B(\mathbf{r}+\mathbf{n}L) + \mathcal{O}\left(\mathrm{e}^{-\sqrt{2}\kappa L}\right)$$

Finite volume mass shift

$$\Delta m_B = \sum_{|\mathbf{n}|=1} \int \mathrm{d}^3 r\, \psi_B^*(\mathbf{r})\, V(\mathbf{r})\, \psi_B(\mathbf{r} + \mathbf{n}L) + \mathcal{O}\left(\mathrm{e}^{-\sqrt{2}\kappa L}\right)$$

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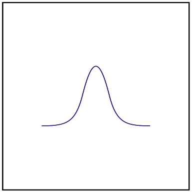

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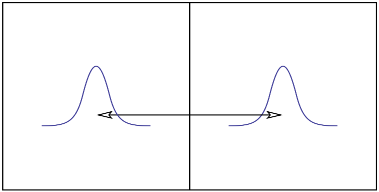

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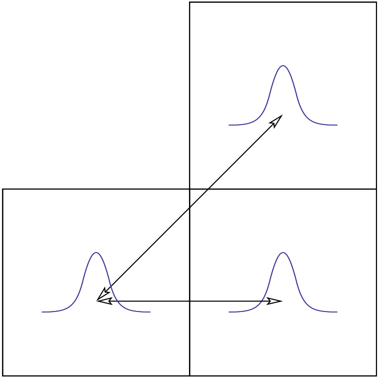

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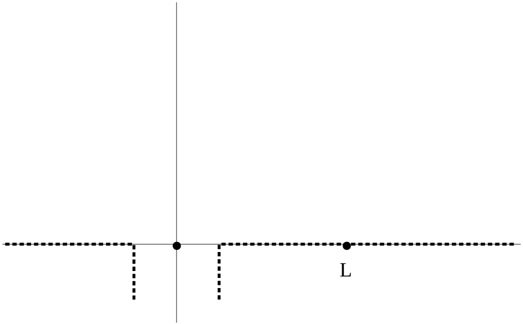

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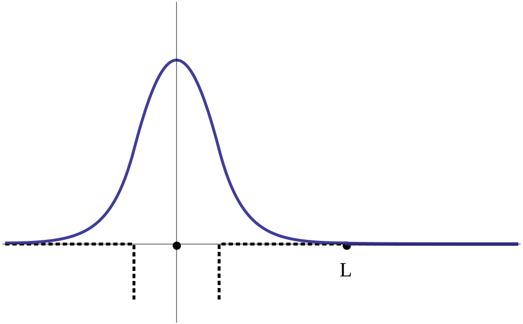

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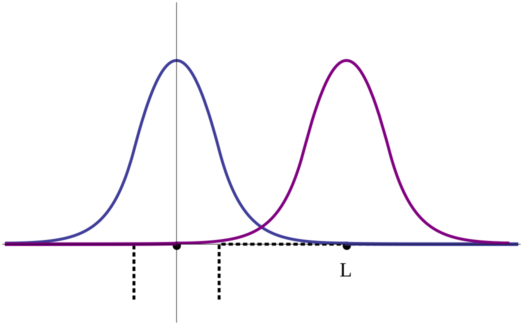

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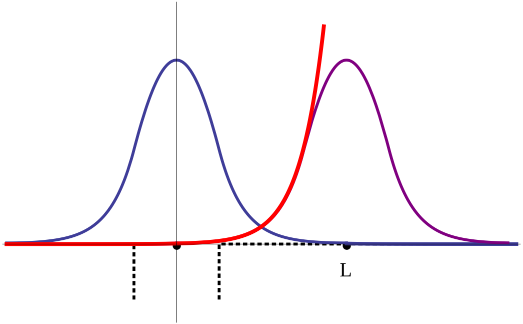

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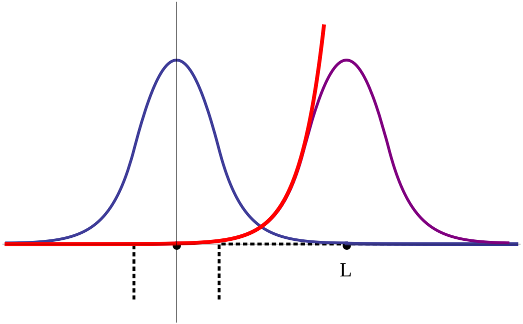
It's all determined by the tail!



Finite volume mass shift

$$\Delta m_B = \sum_{|\mathbf{n}|=1} \int \mathrm{d}^3 r \, \psi_B^*(\mathbf{r} - \mathbf{n}L) \, V(\mathbf{r} - \mathbf{n}L) \, Y_\ell^m(\theta, \phi) \frac{\mathrm{i}^\ell \gamma \hat{h}_\ell^+(\mathrm{i}\kappa r)}{r} + \cdots$$

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Finite volume mass shift

$$\Delta m_B = \sum_{|\mathbf{n}|=1} \int \mathrm{d}^3 r \, \frac{1}{2\mu} \Big[\Delta_r - \kappa^2\Big] \psi_B^*(\mathbf{r} - \mathbf{n}L) \, Y_\ell^m(\theta, \phi) \frac{\mathrm{i}^\ell \gamma \hat{h}_\ell^+(\mathrm{i}\kappa r)}{r} + \cdots$$

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S-waves $\rightarrow \; Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \;\; , \;\; \hat{h}_0^+(\mathrm{i}\kappa r) = \mathrm{e}^{-\kappa r}$

Finite volume mass shift

$$\Delta m_B = \sum_{|\mathbf{n}|=1} \int \mathrm{d}^3 r \, \frac{\gamma}{\sqrt{16\pi\mu}} \left[\Delta_r - \kappa^2\right] \psi_B^*(\mathbf{r} - \mathbf{n}L) \, \frac{\mathrm{e}^{-\kappa r}}{r} + \cdots$$

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$\left[\Delta_r - \kappa^2\right] \dfrac{\mathrm{e}^{-\kappa r}}{r} = -4\pi \delta^{(3)}(\mathbf{r})$ $\rightarrow$ **Green's function**

S-waves

sum just yields a factor six...

$$\Delta m_B^{(0,0)} = -3|\gamma|^2 \, \frac{\mathrm{e}^{-\kappa L}}{\mu L} + \mathcal{O}\left(\mathrm{e}^{-\sqrt{2}\kappa L}\right)$$

Higher partial waves

$$\Delta m_B = \sum_{|\mathbf{n}|=1} \int \mathrm{d}^3 r \, \frac{1}{2\mu} \left[\Delta_r - \kappa^2\right] \psi_B^*(\mathbf{r} - \mathbf{n}L) \, Y_\ell^m(\theta,\phi) \frac{\mathrm{i}^\ell \gamma \hat{h}_\ell^+(\mathrm{i}\kappa r)}{r} + \cdots$$

Lemma

$$Y_\ell^m(\theta,\phi) \, \frac{\hat{h}_\ell^+(\mathrm{i}\kappa r)}{r} = (-\mathrm{i})^\ell \, R_\ell^m \left(-\frac{1}{\kappa} \boldsymbol{\nabla}_r\right) \left[\frac{\mathrm{e}^{-\kappa r}}{r}\right]$$

$R_\ell^m(\mathbf{r}) = r^\ell Y_\ell^m(\hat{\mathbf{r}})$

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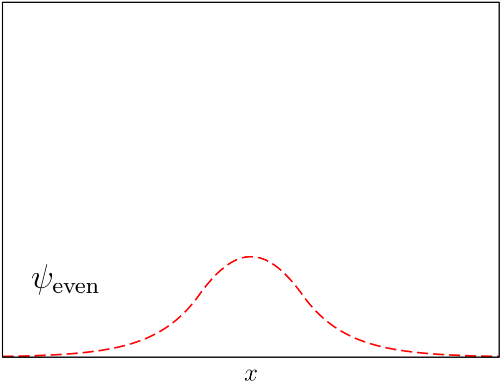
P-waves

$$\Delta m_B^{(1,0)} = \Delta m_B^{(1,\pm 1)} = 3|\gamma|^2 \, \frac{\mathrm{e}^{-\kappa L}}{\mu L} + \mathcal{O}(\mathrm{e}^{-\sqrt{2}\kappa L})$$

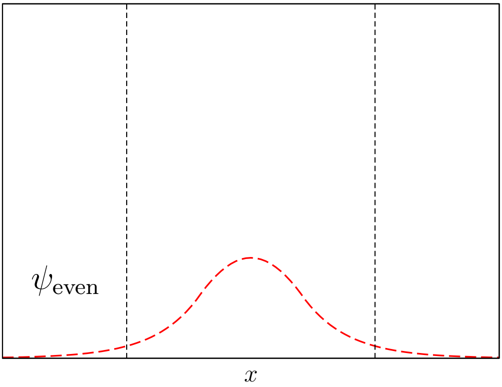
Mass shift for P-wave states exactly reversed compared to S-waves!

Sign of the mass shift



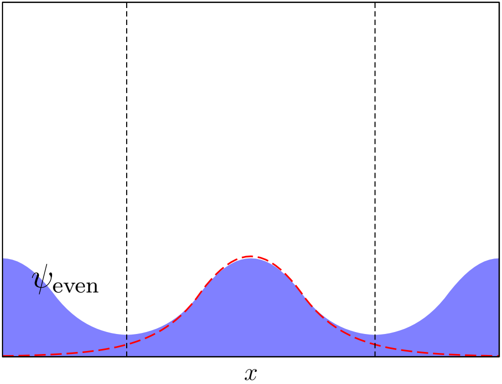
$$\Delta m_B < 0$$

Sign of the mass shift



$$\Delta m_B < 0$$

Sign of the mass shift

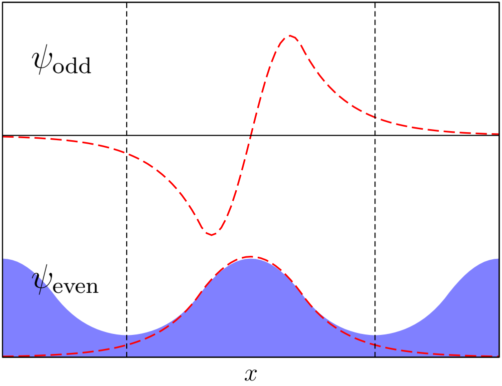


$\Delta m_B < 0$

even parity $\rightarrow$ WF profile relaxed $\rightarrow$ less curvature

$\rightsquigarrow$ **more deeply bound**

Sign of the mass shift



$\Delta m_B > 0$

$\Delta m_B < 0$

even parity $\rightarrow$ WF profile relaxed $\rightarrow$ less curvature

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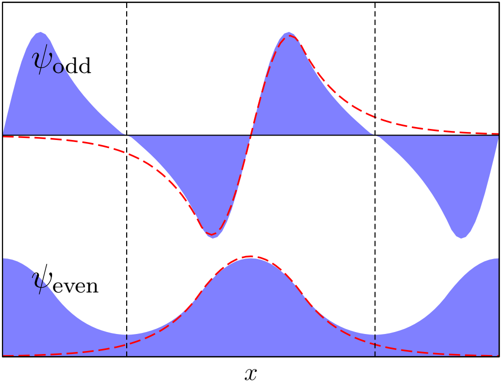
Sign of the mass shift

odd parity → WF profile compressed → more curvature

⇝ **less bound**



$\Delta m_B > 0$

$\Delta m_B < 0$

even parity → WF profile relaxed → less curvature

⇝ **more deeply bound**

Final result

General formula

$$\Delta m_B^{(\ell,\Gamma)} = \alpha\left(\frac{1}{\kappa L}\right) \times |\gamma|^2 \frac{\mathrm{e}^{-\kappa L}}{\mu L} + \mathcal{O}(\mathrm{e}^{-\sqrt{2}\kappa L})$$

ℓ	Γ	$\alpha(x)$
0	A_1^+	-3
1	T_1^-	$+3$
2	T_2^+	$30x + 135x^2 + 315x^3 + 315x^4$
2	E^+	$-1/2\,(15 + 90x + 405x^2 + 945x^3 + 945x^4)$

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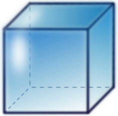
broken symmetry!

rotation group $SO(3)$

$\longrightarrow$



cubic group O

Numerical checks

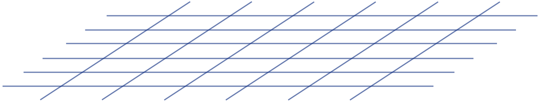
Results can be checked with a very simple calculation...

Lattice Hamiltonian

$$\hat{H}_0 = \sum_{\hat{\mathbf{n}}} \left[\frac{3}{\hat{\mu}} a^\dagger(\hat{\mathbf{n}}) a(\hat{\mathbf{n}}) - \frac{1}{2\hat{\mu}} \sum_{l=1,2,3} \Big(a^\dagger(\hat{\mathbf{n}}) a(\hat{\mathbf{n}} + \hat{\mathbf{e}}_l) + a^\dagger(\hat{\mathbf{n}}) a(\hat{\mathbf{n}} - \hat{\mathbf{e}}_l) \Big) \right]$$

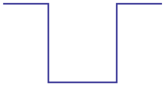
$$\hat{E}(\hat{\mathbf{q}}) = \frac{1}{\hat{\mu}} \sum_{l=1,2,3} (1 - \cos \hat{q}_i) = \frac{1}{2\hat{\mu}} \sum_{l=1,2,3} \hat{q}_l^2 \left[1 + \mathcal{O}(\hat{q}_l^2) \right]$$

lattice units: $\hat{L} = L/a$, $\hat{E} = E \cdot a$, etc. , $a =$ lattice spacing



Numerical checks

- Interaction: $V_{\text{step}}(r) = -V_0\,\theta(R-r)$
- Approximate infinite volume with $L_\infty = 40$



Methods to calculate mass shift

1. $\Delta m_B = E_B(L_\infty) - E_B(L)$ (direct difference)
2. $\Delta m_B = \sum_{|\mathbf{n}|=1} \int \mathrm{d}^3 r\, \psi_B^*(\mathbf{r})\, V(\mathbf{r})\, \psi_B(\mathbf{r}+\mathbf{n}L)$ (overlap integral)
3. $\Delta m_B = \alpha\left(\frac{1}{\kappa L}\right) \cdot |\gamma|^2\, \frac{\mathrm{e}^{-\kappa L}}{\mu L}$ (Green's function)

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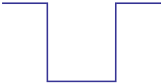

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- Replace

$$\mathrm{e}^{-\hat{\kappa}\hat{L}}/\hat{L} \longrightarrow 4\pi \hat{G}_{\hat{\kappa}}(\hat{L}, 0, 0)$$

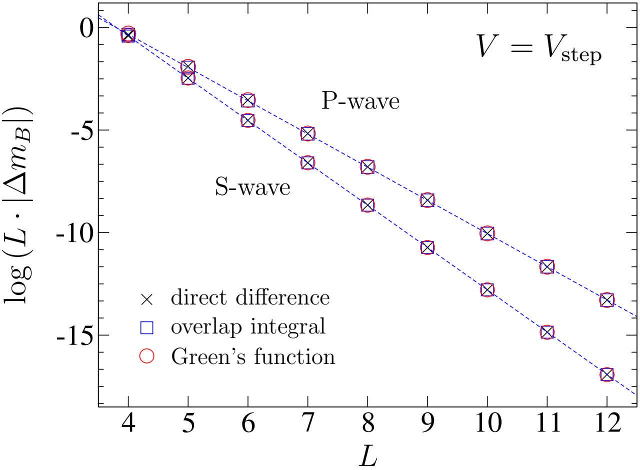
to reduce discretization errors!

Lattice Green's function

$$\hat{G}_{\hat{\kappa}}(\hat{\mathbf{n}}) = \frac{1}{L^3} \sum_{\hat{\mathbf{q}}} \frac{\mathrm{e}^{-\mathrm{i}\hat{\mathbf{q}}\cdot\hat{\mathbf{n}}}}{Q^2(\hat{\mathbf{q}}) + \hat{\kappa}^2}$$

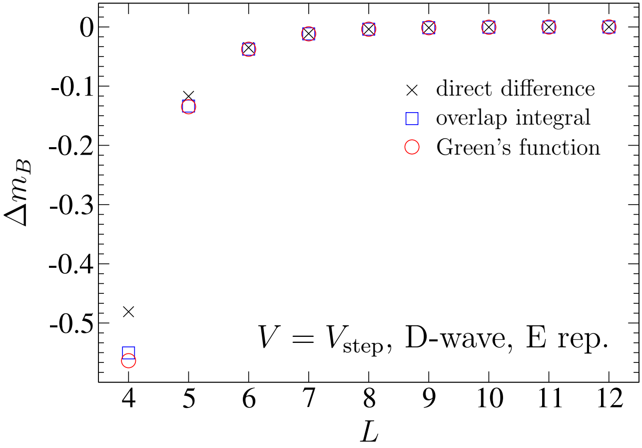
Numerical checks

$$\Delta m_B = \pm 3 \cdot |\gamma|^2 \, \frac{e^{-\kappa L}}{\mu L}$$



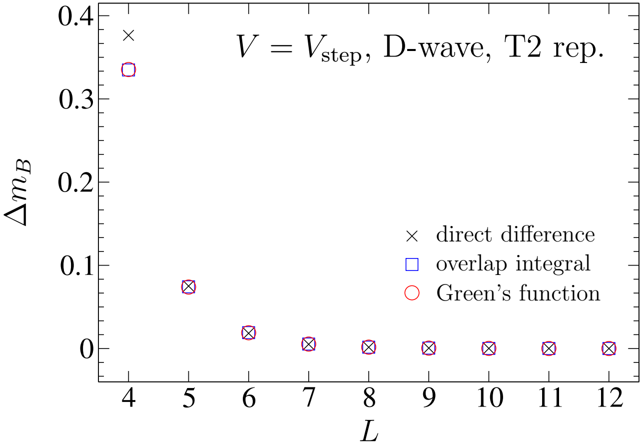
Numerical checks

$$\Delta m_B = \left(\frac{30}{\kappa L} + \cdots\right) \cdot |\gamma|^2 \, \frac{e^{-\kappa L}}{\mu L}$$



Numerical checks

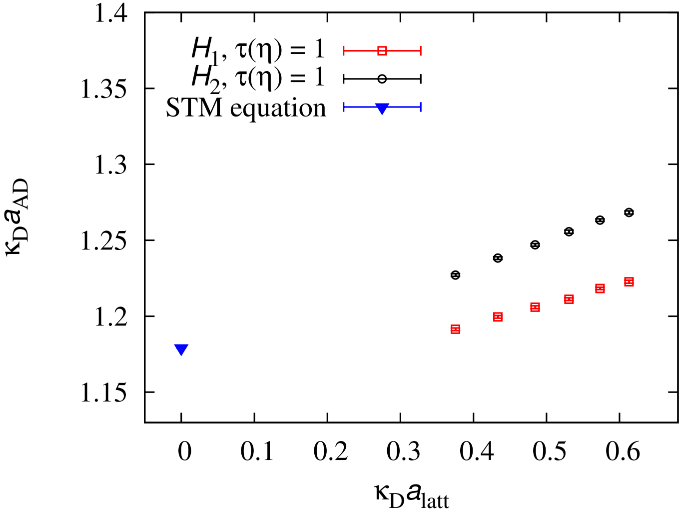
$$\Delta m_B = -{}^1\!/\!{}_2 \left(15 + \frac{90}{\kappa L} + \cdots\right) \cdot |\gamma|^2 \, \frac{\mathrm{e}^{-\kappa L}}{\mu L}$$



Part II

Topological factors in scattering systems

Motivation: atom-dimer scattering



calculation by S. Bour, D. Lee, U.-G. Meißner

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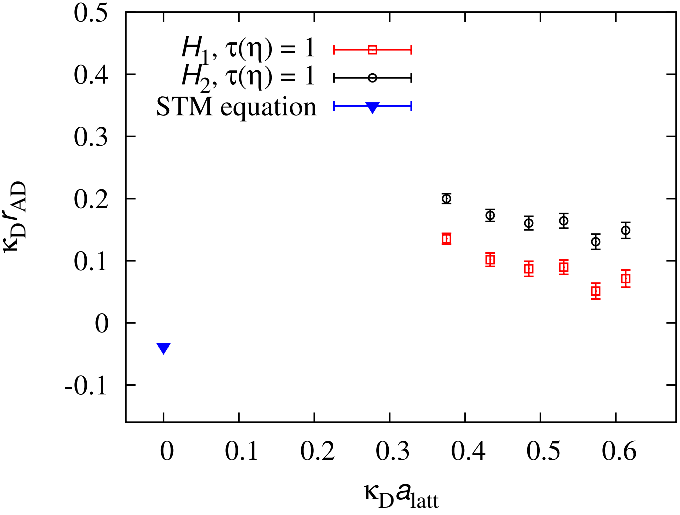
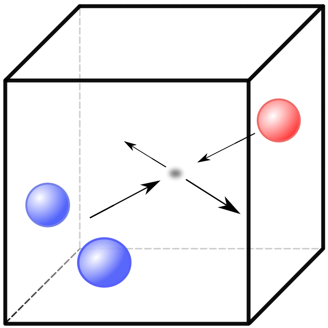


calculation by S. Bour, D. Lee, U.-G. Meißner

Atom–dimer scattering

$$p \cot \delta_0(p) = \frac{1}{\pi L} S(\eta) \quad , \quad \eta = \left(\frac{Lp}{2\pi}\right)^2 \quad , \quad p = p(E(L))$$

$$p \cot \delta_0(p) = -\frac{1}{a_{\mathrm{AD}}} + \frac{r_{\mathrm{AD}}}{2} p^2 + \mathcal{O}(p^4)$$



A part of $E(L)$ is due to the binding of the dimer!

Part II

Topological factors in scattering systems

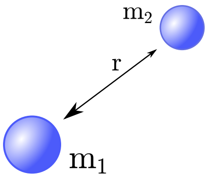
- Motivation ✓
- Moving bound states in a finite volume
- Mass shift for twisted boundary conditions
- Corrections for scattering states
- Conclusion: corrected atom-dimer results

Bound states in moving frames

So far…

considered two-particle state directly in relative coordinates

wavefunction $\psi(\mathbf{r})$, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$



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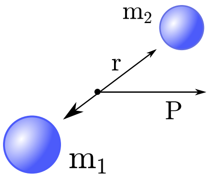
Now

full wavefunction $\Psi(\mathbf{r}_1, \mathbf{r}_2) = \mathrm{e}^{\mathrm{i}\mathbf{P}\cdot\mathbf{R}}\, \psi(\mathbf{r})$

$\mathbf{P}$ = center-of-mass momentum

$\mathbf{R} = \alpha\mathbf{r}_1 + (1-\alpha)\mathbf{r}_2$, $\alpha = \dfrac{m_1}{m_1 + m_2}$



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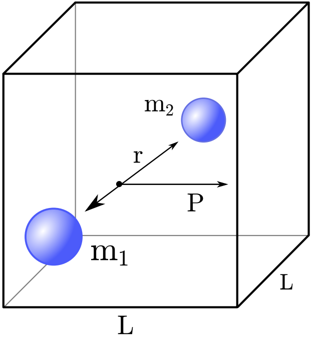
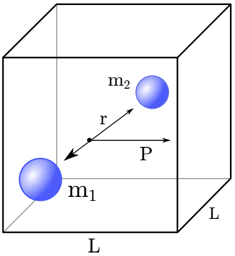


Put system into finite box, impose periodic BC...

Twisted boundary conditions

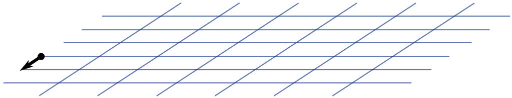


Now $\Psi(\mathbf{r}_1,\mathbf{r}_2)$ has to be periodic!

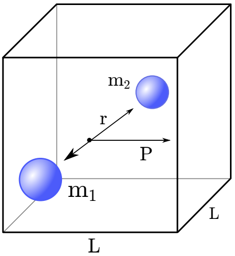
$$\Psi(\mathbf{r}_1{+}\mathbf{n}L,\mathbf{r}_2) = \mathrm{e}^{\mathrm{i}\mathbf{P}\cdot\mathbf{R}}\,\mathrm{e}^{\mathrm{i}\alpha L\mathbf{P}\cdot\mathbf{n}}\,\psi(\mathbf{r}{+}\mathbf{n}L) = \Psi(\mathbf{r}_1,\mathbf{r}_2)$$

$$\rightsquigarrow \psi(\mathbf{r}+\mathbf{n}L) = \mathrm{e}^{-\mathrm{i}\alpha L\mathbf{P}\cdot\mathbf{n}}\,\psi(\mathbf{r})$$

“twisted boundary conditions”



Twisted boundary conditions



Now $\Psi(\mathbf{r}_1, \mathbf{r}_2)$ has to be periodic!

$$\Psi(\mathbf{r}_1+\mathbf{n}L, \mathbf{r}_2) = \mathrm{e}^{\mathrm{i}\mathbf{P}\cdot\mathbf{R}}\,\mathrm{e}^{\mathrm{i}\alpha L\mathbf{P}\cdot\mathbf{n}}\,\psi(\mathbf{r}+\mathbf{n}L) = \Psi(\mathbf{r}_1, \mathbf{r}_2)$$

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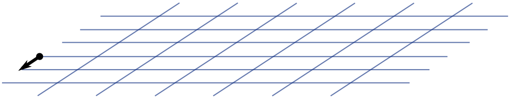

Question

What is the finite-volume mass shift in this case?

Mass shift for twisted boundary conditions

- boundary condition: $\psi(\mathbf{r}+\mathbf{n}L) = \mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}\cdot\mathbf{n}}\,\psi(\mathbf{r})$, $\boldsymbol{\theta} = \alpha L \mathbf{P}$
- new ansatz: $\psi_0(\mathbf{r}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} \psi_B(\mathbf{r}+\mathbf{n}L)\,\mathrm{e}^{\mathrm{i}\boldsymbol{\theta}\cdot\mathbf{n}}$

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$$\rightsquigarrow \Delta m_B = \sum_{|\mathbf{n}|=1} \int \mathrm{d}^3r\,\psi_B^*(\mathbf{r})\,V(\mathbf{r})\,\psi_B(\mathbf{r}+\mathbf{n}L)\,\mathrm{e}^{\mathrm{i}\boldsymbol{\theta}\cdot\mathbf{n}} + \mathcal{O}(\mathrm{e}^{-\sqrt{2}\kappa L})$$

S-wave result

$$\Delta m_B = -|\gamma|^2\,\frac{\mathrm{e}^{-\kappa L}}{\mu L} \times \sum_{\mathbf{n}=\hat{\mathbf{e}}_x,\hat{\mathbf{e}}_y,\hat{\mathbf{e}}_z} \cos(\boldsymbol{\theta}\cdot\mathbf{n}) + \cdots$$

Mass shift for twisted boundary conditions

- boundary condition: $\psi(\mathbf{r}+\mathbf{n}L) = \mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}\cdot\mathbf{n}}\,\psi(\mathbf{r})$, $\boldsymbol{\theta} = \alpha L\mathbf{P}$
- new ansatz: $\psi_0(\mathbf{r}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} \psi_B(\mathbf{r}+\mathbf{n}L)\,\mathrm{e}^{\mathrm{i}\boldsymbol{\theta}\cdot\mathbf{n}}$

$$\rightsquigarrow \Delta m_B = \sum_{|\mathbf{n}|=1} \int \mathrm{d}^3r\,\psi_B^*(\mathbf{r})\,V(\mathbf{r})\,\psi_B(\mathbf{r}+\mathbf{n}L)\,\mathrm{e}^{\mathrm{i}\boldsymbol{\theta}\cdot\mathbf{n}} + \mathcal{O}(\mathrm{e}^{-\sqrt{2}\kappa L})$$

S-wave result

$$\Delta m_B = -|\gamma|^2\,\frac{\mathrm{e}^{-\kappa L}}{\mu L} \times \sum_{\mathbf{n}=\hat{\mathbf{e}}_x,\hat{\mathbf{e}}_y,\hat{\mathbf{e}}_z} \cos(\boldsymbol{\theta}\cdot\mathbf{n}) + \cdots$$

$\sum_{\mathbf{n}} \cos(\boldsymbol{\theta}\cdot\mathbf{n}) = 3$ for $\boldsymbol{\theta} = (0,0,0)$ $\rightarrow$ consistent ✓

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The mass shift vanishes in certain moving frames!

$\rightarrow$ Davoudi & Savage, arXiv:1108.5371

Scattering states

Now consider the scattering of two states A and B...

Scattering wavefunction

$$\langle \vec{r} \,|\Psi_p\rangle = c\sum_{\vec{k}} \frac{e^{i\frac{2\pi\vec{k}}{L}\cdot\vec{r}}}{(2\pi\vec{k}/L)^2 - p^2} \ , \ E_{AB}(p,L) = \frac{\langle\Psi_p|\,\hat{H}\,|\Psi_p\rangle}{\langle\Psi_p\,|\Psi_p\rangle}$$

- $\Delta E^A_{\vec{k}}(L) \equiv E^A_{\vec{k}}(L) - E^A_{\vec{k}}(\infty) = 0$
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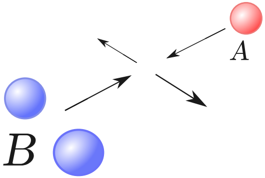

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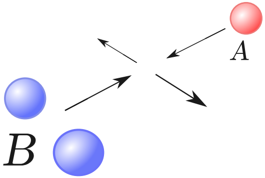


Topological correction factors

Topological volume factor

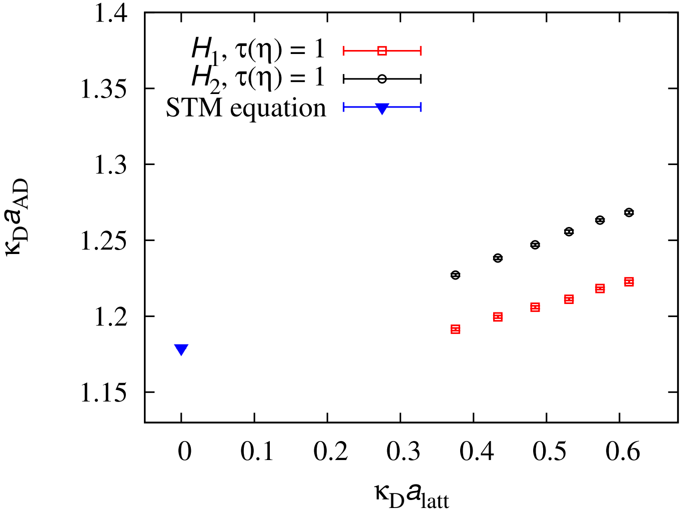
$$\tau(\eta) = \frac{1}{\mathcal{N}} \sum_{\vec{k}} \frac{\sum_{l=1,2,3} \cos(2\pi\alpha k_l)}{3\left(\vec{k}^2 - \eta\right)^2} \quad , \quad \eta = \left(\frac{Lp}{2\pi}\right)^2$$

Final result:

$$E_{AB}(p,L) - E_{AB}(p,\infty) = \tau_A(\eta)\,\Delta E_{\vec{0}}^{A}(L) + \tau_B(\eta)\,\Delta E_{\vec{0}}^{B}(L)$$

Subtract this correction from measured energy levels! $\Rightarrow$

Corrected atom-dimer results



calculation by S. Bour, D. Lee, U.-G. Meißner

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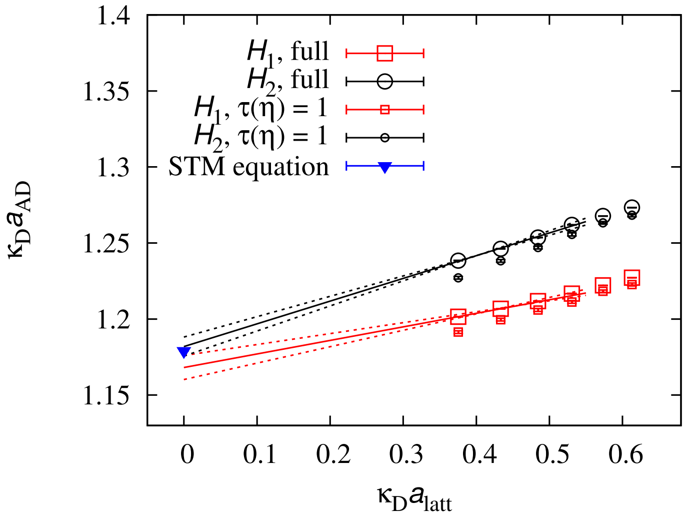

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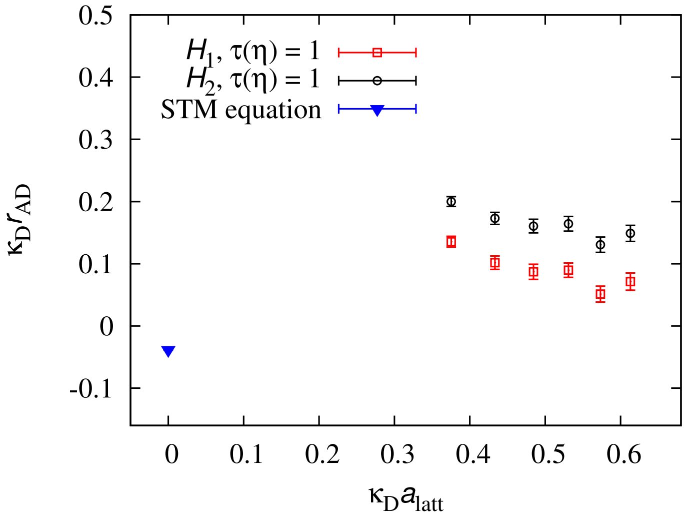

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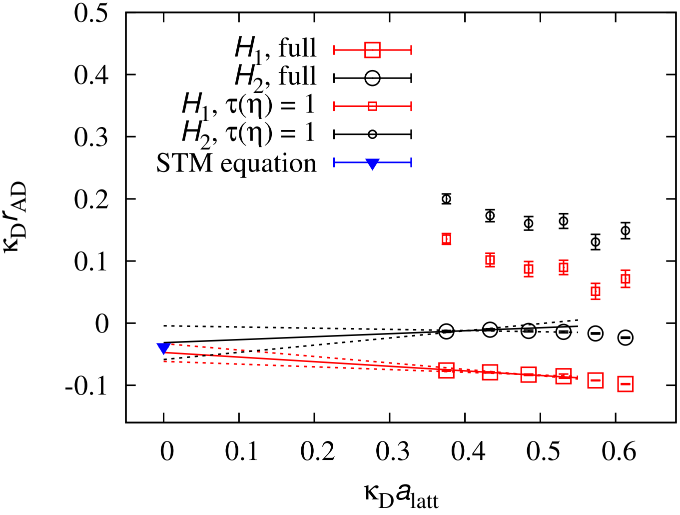

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The End

Summary

- Mass shift can be calculated for bound states with arbitrary ℓ.
- Sign of the shift can be related to parity of the states.
- Predictions can be tested by numerical calculations.
- Mass shift of composite particles has to be corrected for in scattering calculations.

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Thanks for your attention!