KAONIC DEUTERIUM
theory status report

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$\bar{K}N$ scattering lengths

- Fundamental quantities to test our understanding of the low energy $SU(3)$ meson-baryon dynamics
- Experiment at DAΦNE:
  - SIDDHARTA: energy shift and width of the $1s$ level for kaonic hydrogen

$\rightarrow$ extract a_{K^-p} from modified Deser-type formula

Meißner, Raha, Rusetsky (2004)

..., which does <u>not</u> determine $\{a_I | I = 0, 1\}$

$\rightarrow$ use as an (additional) input for chiral unitary models to predict a_0 and a_1

Ikeda, Hyodo, Weise (2012) Mai, Meißner (2012)

$\bar{K}N$ scattering lengths

- Experiments at DAΦNE:
  - SIDDHARTA: energy shift and width of the $1s$ level for kaonic hydrogen
  - SIDDHARTA-2(under preparation): $1s$ level transition of kaonic deuterium
- Extraction of the antikaon-nucleon scattering lengths:
  1) threshold scattering amplitudes a_{K^-p} and A_{K^-d} via modified Deser-type formula

     Meißner, Raha, Rusetsky (2006)
  2) relate a_{K^-p} and A_{K^-d} to the S-wave $\bar{K}N$ scattering lengths

$\bar{K}d$ scattering

- Three-body Faddeev equation:
  - allows for an analysis of $\bar{K}NN$ system
  - requires knowledge of the NN as well as $\bar{K}N$ potential
    $\Rightarrow$ no explicit relation btw. A_{K^-d} and a_I
- Multiple-scattering series — Kamalov, Oset, Ramos (2001)
  - poor convergence of the expansion
  - resummation of the series $\hat{=}$ fixed-center-approximation (FCA) of Faddeev Equations ($m_N \to \infty$) $\Rightarrow$ *Brueckner-type formula*:

$$R_{K^-d} \propto \int d^3r \Psi^2(r) \frac{\tilde{a}_p + \tilde{a}_n + (2\tilde{a}_p\tilde{a}_n - b_x^2)/r - 2b_x^2\tilde{a}_n/r^2}{1 - \tilde{a}_p\tilde{a}_n/r^2 + b_x^2\tilde{a}_n/r^3}$$

$\tilde{a}_i = (1 + \frac{M_K}{m_N})$, $b_x := \tilde{a}_x/\sqrt{1 + \tilde{a}_0/r}$
$a_x := a_{K^-p\to\bar{K}^0n}$, $a_p := a_{K^-p\to\bar{K}^-p}$, $a_0 := a_{\bar{K}^0n\to\bar{K}^0n}$,
$a_n := a_{K^-n\to K^-n}$

Recoil corrections

- $\Lambda(1405)$ located close to the threshold $\Rightarrow$ higher terms in effective-range expansion (?)
- $\xi := M_K/m_N \approx 0.5 \neq 0 \Rightarrow$ consider *recoil corrections*

  Baru, Epelbaum, Rusetsky (2009)

  - systematic calculation of corrections for double scattering
  - non-relativistic framework
  - systematic expansion in powers of ξ (actually $\sqrt{\xi} \sim 0.7$)
  - one retardation insertion yields a change of $\sim 15\%$

$\Rightarrow$ Our goal:

- extend this framework to multiple scattering diagrams
- estimate the size of recoil corrections in multiple scattering
- estimate convergence in parameter ξ

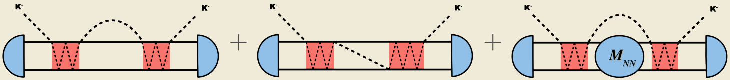
Idea

- assume a single interaction channel, specified by scattering length a, then

$$R_{K^-d} = a + a^2 D + a^3 D^2 + \ldots = a + a^2(\tilde{D} + \frac{1}{r}) + a^3(\tilde{D} + \frac{1}{r})^2 + \ldots$$
$$= a + \frac{a^2}{r} + \frac{a^3}{r^2} + \ldots + (a^2 + 2\frac{a^3}{r} + \ldots)\tilde{D} + (a^3 + \ldots)\tilde{D}^2 + \ldots$$
$$= R_{st} + R^{(1)} + R^{(2)} + \ldots,$$

where D and $\tilde{D} := D - \frac{1}{r}$ are the full and retarded kaon propagators.

$\Rightarrow$ in the realistic case, <u>for one retardation</u> we have to compute:

$R^{(1)} =$ 

One retardation

- after some combinatorial analysis...
  - $R_{st} = \frac{1}{8\pi} \int d^3 r \Psi(r)^2 X_{NN}(r)$
  - $R_{0/1} = \frac{1}{2} \frac{\xi}{(1+\xi)} \int \frac{d^3p\, d^3l}{(2\pi)^6} \frac{1}{l^2} \frac{l^2/2 - b_p^2}{l^2(1+\xi/2)+\xi b_p^2} \Phi_{\pm}(p,l)$
  - $R_{NN} = \frac{\xi}{4m_N} \int \frac{d^3p\, d^3l\, d^3q}{(2\pi)^9} \frac{M_{NN}(p,q,l)\Phi_{NN}(p,q,l)}{(l^2(1+\xi/2)+b_p^2\xi)(l^2(1+\xi/2)+b_q^2\xi)}$
  - $\Delta R^{st}_{0/1} = (\pm) \frac{\xi}{1+\xi} \frac{1}{8\pi} \int d^3 r \frac{1}{r} (\Psi(r) X_{\pm}(r))^2$

...with

- $A_i = \frac{8\pi}{1+\xi/2} R_i$
- $b_p^2 := 2(p^2 + m_N \epsilon_d)$
- $X_{NN}(r) = \sqrt{2} X_+(r) = \frac{r(\tilde{a_0}(4\tilde{a}1+r)+3\tilde{a}_1 r)}{\tilde{a}_0(r-2\tilde{a}_1)+r(2r-\tilde{a}_1)}$
- $X_-(r) = \frac{\sqrt{6}(\tilde{a}_0-\tilde{a}_1)r^2}{\tilde{a}_0(4\tilde{a}_1-2r)+2(\tilde{a}_1-2r)r}$
- $\Phi_\pm(p,l) = \widetilde{X_\pm \Psi}(p+l/2)\left(\widetilde{X_\pm \Psi}(p+l/2) \pm \widetilde{X_\pm \Psi}(p-l/2)\right)$
- $\Phi_{NN}(p,q,l) = \widetilde{X_{NN}\Psi}(p+l/2)\widetilde{X_{NN}\Psi}\,(q+l/2)$
- $\widetilde{X_i \Psi}(P) := \int d^3 r X_i(r) \Psi(r)\, e^{iPr}$
- $M_{NN}(p,q,l) = V(p,q) + \frac{1}{4m_N} \int \frac{d^3 p'}{(2\pi)^3} \frac{V_(p,p')M_{NN}(p',q,l)}{p'^2+m_n\epsilon_d+l^2(\frac{1+\xi/2}{2\xi})}$

One retardation - full contribution

- Compare to the outcome of Faddeev equations (FE) Shevchenko (2012)
  - for $a_1 = -1.62 + i0.78$ fm, $a_0 = 0.18 + i0.68$ fm

	PEST Zankel et al. (1983)	**TSA-A** Doleschall	**TSA-B** Doleschall
A_{st}	$-1.549 + i1.245$	$-1.515 + i1.207$	$-1.503 + i1.194$
ΔA_0^{st}	$-0.111 + i0.344$	$-0.115 + i0.335$	$-0.116 + i0.334$
ΔA_1^{st}	$+0.128 - i0.219$	$+0.123 - i0.206$	$+0.119 - i0.201$
A_0	$-0.287 + i0.954$	$-0.274 + i0.890$	$-0.275 + i0.877$
A_1	$-0.125 + i0.173$	$-0.123 + i0.156$	$-0.120 + i0.149$
A_{NN}	$+0.364 - i1.208$	$+0.362 - i1.118$	$+0.365 - i1.097$
A_{full}	$-1.579 + i1.289$	$-1.541 + i1.265$	$-1.530 + i1.256$
FE	$-1.580 + i1.130$	$-1.570 + i1.100$	$-1.570 + i1.110$

One retardation - full contribution

- ... and for $a_1 = -1.60 + i0.67$ fm, $a_0 = -0.004 + i0.57$ fm

	PEST Zankel et al. (1983)	**TSA-A** Doleschall	**TSA-B** Doleschall
A_{st}	$-1.485 + i1.078$	$-1.450 + i1.047$	$-1.437 + i1.037$
ΔA_0^{st}	$-0.103 + i0.288$	$-0.105 + i0.282$	$-0.105 + i0.281$
ΔA_1^{st}	$+0.068 - i0.217$	$+0.066 - i0.202$	$+0.062 - i0.197$
A_0	$-0.277 + i0.700$	$-0.264 + i0.747$	$-0.261 + i0.737$
A_1	$-0.076 + i0.168$	$-0.080 + i0.150$	$-0.077 + i0.143$
A_{NN}	$+0.352 - i1.006$	$+0.341 - i0.933$	$+0.338 - i0.917$
A_{full}	$-1.521 + i1.111$	$-1.491 + i1.090$	$-1.479 + i1.084$
FE	$-1.510 + i0.990$	$-1.500 + i0.970$	$-1.490 + i0.980$

Good news: Real part is closer to FE results after an insertion of one retardation

Unclear: Imaginary part moves further away

Expansion in powers of ξ

- recipe worked out for arbitrary Feynman diagrams:

  Baru, Epelbaum, Rusetsky (2009)

  1) Identify the momentum scales, e.g. small scale λ, large scale Λ.
  2) Expand the integrand $f(\lambda, q, \Lambda)$ in the low-, high- and intermediate momentum regime, i.e $\lambda \sim q \ll \Lambda$, $\lambda \ll q \sim \Lambda$ and $\lambda \ll q \ll \Lambda$.
  3) $\int_q f(\lambda, q, \Lambda) = \int_q f_l(\lambda, q, \Lambda) - \int_q f_i(\lambda, q, \Lambda) + \int_q f_h(\lambda, q, \Lambda)$.

- take a look on $R_{0/1}$:

$$
\begin{aligned}
R_{0/1} &= \frac{1}{2}\frac{\xi}{(1+\xi)} \int \frac{d^3p\, d^3l}{(2\pi)^6}\, \frac{1}{l^2}\, \frac{l^2/2 - b_p^2}{l^2(1+\xi/2) + \xi b_p^2} \Phi_\pm(p,l) \\
&= \frac{\xi}{2(2+\xi)} \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{l^2} \left(\frac{1}{1+\xi} - \frac{2b_p^2}{l^2(1+\xi/2) + \xi b_p^2} \right) \Phi_\pm(p,l) \\
&= \frac{\xi}{(1+\xi)(2+\xi)} J_0 - \frac{\xi}{(1+\xi/2)^2} T_{1/0}
\end{aligned}
$$

Expansion in powers of ξ

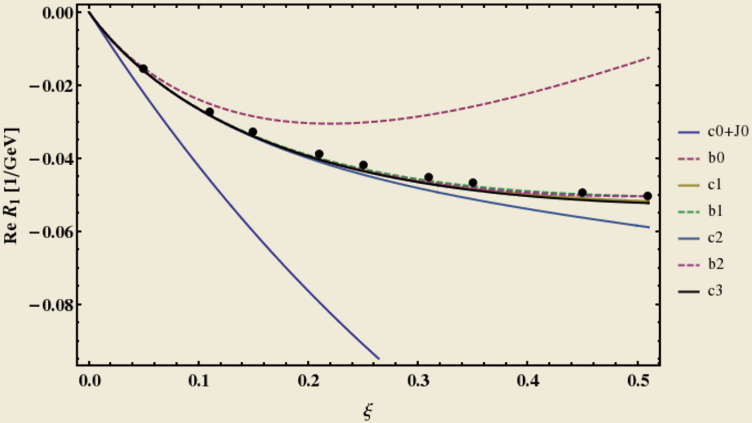
- low-momentum: $T^l_\pm = \int \frac{d^3p d^3q}{(2\pi)^6} \frac{b^2}{l^2} \left(\Phi^\pm_0 + \Phi^\pm_2 l^2 + \Phi^\pm_4 l^4 + ... \right) \frac{1}{l^2 + \xi \tilde{b}^2}$
- high-momentum: $T^h_\pm = \int \frac{d^3p d^3q}{(2\pi)^6} \frac{b^2}{l^2} \Phi_\pm(p,l) \left(1 - \frac{\xi \tilde{b}^2}{l^2} + \frac{\xi^2 \tilde{b}^4}{l^4} + ... \right)$
- intermediate: $T^i_\pm = \int \frac{d^3p d^3q}{(2\pi)^6} \frac{b^2}{l^2} \left(\Phi^\pm_0 + \Phi^\pm_2 l^2 + \Phi^\pm_4 l^4 + .. \right) \left(1 - \frac{\xi \tilde{b}^2}{l^2} + \frac{\xi^2 \tilde{b}^4}{l^4} + .. \right)$

for $\tilde{b}^2 = b^2/(1+\xi/2)$, $\tilde{\xi} = \xi/(1+\xi/2)$ and $\Phi^\pm_{2n} = \frac{1}{(2n+1)2n!} \Delta^n_l \Phi_\pm(p,l)\Big|_{l=0}$

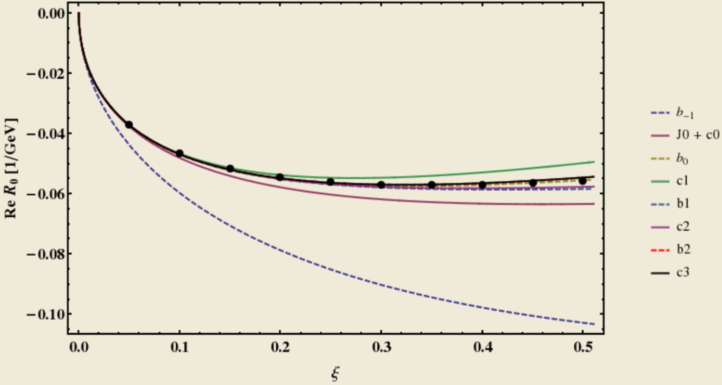
$$R_0 = \frac{2}{1+\xi} \tilde{\xi} \tilde{\boldsymbol{J}}_{\boldsymbol{0}} - \frac{2}{2+\xi} \left(\tilde{\xi} \tilde{\boldsymbol{c}}_{\boldsymbol{0}} - \tilde{\xi}^2 \tilde{\boldsymbol{c}}_{\boldsymbol{1}} + \tilde{\xi}^3 \tilde{\boldsymbol{c}}_{\boldsymbol{2}} + ... + \tilde{\xi}^{1/2} \tilde{\boldsymbol{b}}_{\boldsymbol{-1}} - \tilde{\xi}^{3/2} \tilde{\boldsymbol{b}}_{\boldsymbol{0}} + \tilde{\xi}^{5/2} \tilde{\boldsymbol{b}}_{\boldsymbol{1}} - ... \right)$$

$$R_1 = \frac{2}{1+\xi} \tilde{\xi} \boldsymbol{J}_{\boldsymbol{0}} - \frac{2}{2+\xi} \left(\tilde{\xi} \boldsymbol{c}_{\boldsymbol{0}} - \tilde{\xi}^2 \boldsymbol{c}_{\boldsymbol{1}} + \tilde{\xi}^3 \boldsymbol{c}_{\boldsymbol{2}} + ... + \qquad - \tilde{\xi}^{3/2} \boldsymbol{b}_{\boldsymbol{0}} + \tilde{\xi}^{5/2} \boldsymbol{b}_{\boldsymbol{1}} - ... \right)$$

Results of the ξ-expansion: R_1

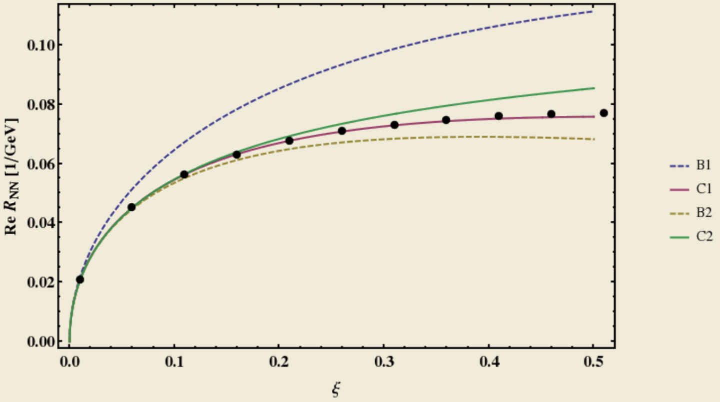


Results of the ξ-expansion: R_0



Results of the ξ-expansion: R_{NN}

- $R_{NN} = \frac{1}{1+\xi/2}\left(\tilde{\xi}\boldsymbol{C_1} + \tilde{\xi}^2\boldsymbol{C_2} + ... + \tilde{\xi}^{1/2}\boldsymbol{B_1} + \tilde{\xi}^{3/2}\boldsymbol{B_2} + ...\right)$



Convergence of the ξ-expansion

Good news: R_0, R_1, R_{NN} converge after a few orders of $\tilde{\xi}$

- with ξ as an expansion parameter more orders are required
- R_{NN} and R_0 cancel at order $\xi^{1/2}$ to a large amount

Puzzling: the leading order corrections in $\tilde{\xi}$ are large, but the full correction of one retardation is small compared to the static part.

Conclusion

Done:

- one-retarded block inserted into multiple scattering diagrams
- integrals are carried out and computed for various "realistic" NN potentials
  - \+ Real part is closer to the result of FE after an insertion of one retardation
  - − Imaginary part is further off this result compared to static part
- expansion of all contributions in $\xi = M_K/m_N$ is performed
  - \+ Separate contributions converge to the full result
  - − Large cancellations between first orders in $\tilde{\xi}$

In preparation:

! **Estimate the size of the double insertion** !

THANK YOU FOR YOUR ATTENTION!

SPARES (R_0 coefficients)

- $\tilde{\boldsymbol{J}}_{\boldsymbol{0}} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3l}{(2\pi)^3}\frac{1}{l^2}\Phi^+(p,l) = \tilde{\boldsymbol{c}}_{\boldsymbol{-1}},\ \tilde{\boldsymbol{b}}_{\boldsymbol{n}} = \frac{1}{2(2\pi)^3}\int dp\ p^2 b^{3+2n}\Phi^+_{2n+2}$
- $\tilde{\boldsymbol{c}}_{\boldsymbol{0}} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3l}{(2\pi)^3}\frac{b^2}{l^4}\Phi^+(p,l)$

  $\tilde{\boldsymbol{c}}_{\boldsymbol{1}} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3l}{(2\pi)^3}\frac{b^4}{l^6}(\Phi^+(p,l) - l^2\Phi^+_2)\ \ldots$

$$\tilde{\boldsymbol{c}}^{\boldsymbol{dim.reg.}}_{\boldsymbol{n-1}} = \int \frac{d^3p}{(2\pi)^3}\frac{d^3l}{(2\pi)^3}\frac{b^{2n}}{l^{2(n+1)}}\Phi^+(p,l)$$

$$= \frac{1}{2^{n+1}\sqrt{\pi}}\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{q=0}^{k}\binom{n}{m}\binom{m}{k}\binom{k}{q}(-1)^{n-m-k}\frac{\Gamma(-m-1+3/2)}{\Gamma(m+1)}\gamma^{2n-2k}2^{-k} \quad \times$$

$$\times \int drr^{2m+1}\left(\Delta_r^q X_+(r)\Psi(r)\right)\left(\Delta_r^{k-q}X_+(r)\Psi(r)\right)$$

$\Rightarrow$ $\tilde{\boldsymbol{b}}_{\boldsymbol{-1}} = (+0.203040 - i\,0.822322)\ \mathrm{GeV}^{-1}$ $\quad \tilde{\boldsymbol{c}}_{\boldsymbol{0}} = (-0.089782 + i\,0.423442)\ \mathrm{GeV}^{-1}$

$\tilde{\boldsymbol{b}}_{\boldsymbol{0}} = (+0.038945 - i\,0.033744)\ \mathrm{GeV}^{-1}$ $\quad \tilde{\boldsymbol{c}}_{\boldsymbol{1}} = (+0.044442 - i\,0.253202)\ \mathrm{GeV}^{-1}$

$\tilde{\boldsymbol{b}}_{\boldsymbol{1}} = (+0.106402 - i\,0.454869)\ \mathrm{GeV}^{-1}$ $\quad \tilde{\boldsymbol{c}}_{\boldsymbol{2}} = (-0.013891 + i\,0.126781)\ \mathrm{GeV}^{-1}$

$\tilde{\boldsymbol{b}}_{\boldsymbol{2}} = (+0.094596 - i\,0.186672)\ \mathrm{GeV}^{-1}$ $\quad \tilde{\boldsymbol{c}}_{\boldsymbol{3}} = (+0.002462 - i\,0.066183)\ \mathrm{GeV}^{-1}$

$\tilde{\boldsymbol{J}}_{\boldsymbol{0}} = (+0.080238 - i\,0.260227)\ \mathrm{GeV}^{-1}$

SPARES (R_1 coefficients)

- $\boldsymbol{J_0} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3l}{(2\pi)^3}\frac{1}{l^2}\Phi^-(p,l) = \boldsymbol{c_{-1}}$, $\boldsymbol{b_n} = -\frac{1}{2}\frac{1}{(2\pi)^3}\int dp\; p^2 b^{3+2n}\Phi^-_{2n+2}$
- $\boldsymbol{c_0} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3l}{(2\pi)^3}\frac{b^2}{l^4}\Phi^-(p,l)$

  $\boldsymbol{c_1} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3l}{(2\pi)^3}\frac{b^4}{l^6}(\Phi^-(p,l) - l^2\Phi^-_2)\ldots$

$$\boldsymbol{c_{n-1}^{dim.reg.}} = \int \frac{d^3p}{(2\pi)^3}\frac{d^3l}{(2\pi)^3}\frac{b^{2n}}{l^{2(n+1)}}\Phi^-(p,l)$$

$$= -\frac{\sqrt{\pi}}{2^n}\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{q=0}^{k}\binom{n}{m}\binom{m}{k}\binom{k}{q}(-1)^{n-m-k}\frac{\Gamma(-m-1+3/2)}{\Gamma(m+1)}\gamma^{2n-2k}2^{-k} \quad\times$$

$$\times\int drr^{2m+1}\left(\Delta_r^q X_-(r)\Psi(r)\right)\left(\Delta_r^{k-q}X_-(r)\Psi(r)\right)$$

$\Rightarrow$ $\boldsymbol{b_0} = (-1.307530 + i\,0.638492)$ GeV^{-1} $\boldsymbol{c_0} = (+0.773933 - i\,0.372541)$ GeV^{-1}
$\boldsymbol{b_1} = (+0.032291 + i\,0.049485)$ GeV^{-1} $\boldsymbol{c_1} = (+0.317277 - i\,0.124332)$ GeV^{-1}
$\boldsymbol{b_2} = (-0.497011 + i\,0.236824)$ GeV^{-1} $\boldsymbol{c_2} = (-0.596553 + i\,0.249037)$ GeV^{-1}
$\boldsymbol{J_0} = (-0.328491 + i\,0.193019)$ GeV^{-1} $\boldsymbol{c_3} = (-0.170478 + i\,0.065114)$ GeV^{-1}

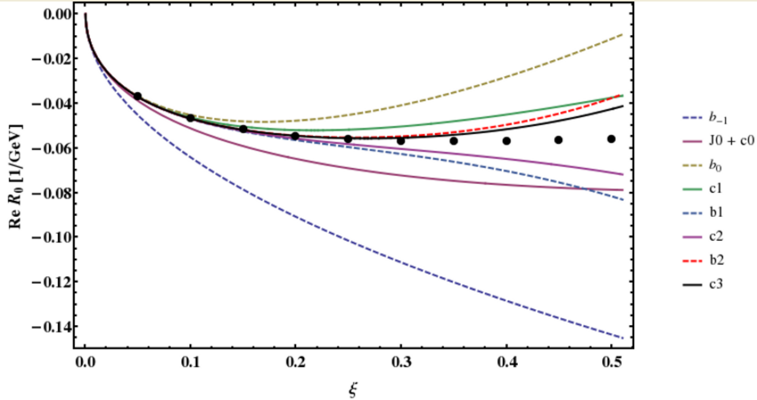
SPARES (R_{NN} coefficients)

- $V(p,q) = \frac{\lambda}{(p^2+\beta^2)(q^2+\beta^2)} \quad \Rightarrow \quad M_{NN}(p,q,l) = \frac{V(p,q)}{1-A(l)}$
- $A(l) = \frac{\lambda}{8m_N} \int \frac{d^3p}{(2\pi)^3} \frac{1}{(p^2+\beta^2)^2(p^2-E(l)m_N-i\epsilon)} = \frac{\lambda(2\beta\sqrt{-E(l)m_N}+\beta^2-E(l)m_N)}{32\pi m_N\beta(\beta^2+E(l)m_N)^2}$
- $T^{low}_{NN} = \frac{1}{4}\int \frac{d^3l}{(2\pi)^3} \frac{\Phi_0^{NN}+l^2\Phi_2^{NN}+l^4\Phi_4^{NN}+\ldots}{(1-A(l))(l^2+b_p^2\frac{\xi}{1+\xi/2})(l^2+b_q^2\frac{\xi}{1+\xi/2})}$
- $T^{high}_{NN} = \int \frac{d^3l}{(2\pi)^3} \frac{\Phi(p,q,l)}{l^4}(1-\frac{b_q^2}{l^2}\frac{\xi}{1+\xi/2}+\ldots)(1-\frac{b_p^2}{l^2}\frac{\xi}{1+\xi/2}+\ldots)$
  $(1+\frac{\lambda}{l^2 16 m_N\beta\pi}\xi+(..)\xi^{3/2}\ldots)$
- $T^{int}_{NN} = \int \frac{d^3l}{(2\pi)^3} \frac{\Phi_0^{NN}+l^2\Phi_2^{NN}+l^4\Phi_4^{NN}+\ldots}{l^4}(1-\frac{b_q^2}{l^2}\frac{\xi}{1+\xi/2}+\ldots)$
  $(1-\frac{b_p^2}{l^2}\frac{\xi}{1+\xi/2}+\ldots)(1+\frac{\lambda}{l^2 16 m_N\beta\pi}\xi+(..)\xi^{3/2}\ldots)$

$$\boldsymbol{B_1} = (+0.88083 - i\,2.96966)\mathrm{GeV}^{-1} \quad \boldsymbol{C_1} = (-0.44499 + i\,0.64918)\mathrm{GeV}^{-1}$$
$$\boldsymbol{B_2} = (-0.14990 + i\,1.03120)\mathrm{GeV}^{-1} \quad \boldsymbol{C_2} = (+0.53600 - i\,1.25200)\mathrm{GeV}^{-1}$$

- CHECK - without resummation ($X_+ = 1$): $2\boldsymbol{B_1} = \tilde{\boldsymbol{b}}_{\boldsymbol{-1}} = 0.0231$. Full cancellation of R_{NN} and R_0 at order $\xi^{1/2}$.

SPARES (R_0 strict expansion in ξ)



SPARES (Convergence of the ξ-expansion)